By— Dr. Dharmendra Kr. Sharma.

B.Sc Sem – IV

MJC – 02

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TOPIC — Euler's theorem

Que :— State and prove that Euler's theorem for function of two variable of degree n.

Proof :- Since $u(x,y)$ be a homogeneous function x & y of degree n then.

$$u = x^n f(y/x)$$

$$\frac{\partial u}{\partial x} = n x^{n-1} f(y/x) + x^n f'(\tfrac{y}{x})\left(\frac{-y}{x^2}\right)$$

$$x \cdot \frac{\partial u}{\partial x} = n x^n f(y/x) - x^{n-1} f'(y/x) \cdot y$$

Again $$\frac{\partial u}{\partial y} = 0 + x^n f'(y/x) \cdot \frac{1}{x}$$

$$\Rightarrow \quad y \cdot \frac{\partial u}{\partial y} = 0 + x^{n-1} f'(y/x) \cdot y$$

$$x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = n x^n \cdot f(y/x) - \cancel{x^{n-1} y\, f'(y/x)} + \cancel{x^{n-1} f'(y/x)\, y}$$

$$= n x^n f(y/x)$$

$$= nu$$

$$x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = nu$$

Now $$x \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y \frac{\partial^2 u}{\partial x \cdot \partial y} = n \cdot \frac{\partial u}{\partial x}$$

$$x \frac{\partial^2 u}{\partial x^2} + y \frac{\partial^2 u}{\partial x \partial y} = (n-1) \frac{\partial u}{\partial x}$$

$$x^2 \frac{\partial^2 u}{\partial x^2} + xy \frac{\partial^2 u}{\partial x \partial y} = (n-1) x \frac{\partial u}{\partial x} \quad \text{— ①}$$

Again

$$x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = nu$$ (2)

$$\Rightarrow x \cdot \frac{\partial^2 u}{\partial x \cdot \partial y} + y \frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y} = n \cdot \frac{\partial u}{\partial y}$$

$$\Rightarrow xy \frac{\partial^2 u}{\partial x \partial y} + xy \frac{\partial u}{\partial y} = (n) xy \cdot \frac{\partial u}{\partial y} \quad \text{— (2)}$$

Adding (1) and (2) $\Rightarrow$

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2}$$

$$= (n-1)\left(x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y}\right)$$

$$= n(n-1)u$$ Prooved

Que (2) :- If $y^{\frac{1}{m}} + y^{-\frac{1}{m}} = 2x$ prove that $(x^2-1)y_2 + xy_1 - m^2 y = 0$

Sol'n :- we have $y^{\frac{1}{m}} + y^{-\frac{1}{m}} = 2x$ — (1)

Differentiating w.r.to x, we get

$$\frac{1}{m} \cdot y^{\frac{1}{m}-1} y_1 + \left(-\frac{1}{m}\right) y^{-\frac{1}{m}-1} y_1 = 2$$

or $\frac{1}{m} y^{-1} \cdot y_1 \left(y^{\frac{1}{m}} - y^{-\frac{1}{m}}\right) = 2$

or $y_1 \left(y^{\frac{1}{m}} - y^{-\frac{1}{m}}\right) = 2my$

Squaring both sides, we get

$$y_1^2 \left(y^{\frac{1}{m}} - y^{-\frac{1}{m}}\right)^2 = 4m^2 y^2$$

or $y_1^2 \left[\left(y^{\frac{1}{m}} + y^{-\frac{1}{m}}\right)^2 - 4 y^{\frac{1}{m}} y^{-\frac{1}{m}}\right] = 4m^2 y^2$

or $y_1^2 \left[(2x)^2 - 4\right] = 4m^2 y^2$, from (1)

or $y_1^2 (x^2 - 1) = m^2 y^2$

Again differentiating w.r.to x, we get

$$(x^2-1) \cdot \frac{d}{dx}(y_1^2) + y_1^2 \cdot \frac{d}{dx}(x^2-1) = m^2 \frac{d}{dx}(y^2)$$

or $(x^2-1) 2y_1 y_2 + y_1^2 (2x) = m^2 \cdot 2y y_1$

dividing both sides by $2y_1$, we get

$$(x^2-1) y_2 + xy_1 - m^2 y = 0$$